The rank of a matrix The rank of a matrix A is the number of leading 1's in rref(A), denoted rank(A).

Example 1 Consider a system of n linear equations with m variables, which has a coefficient matrix A of size $n \times m$. Show that

(1) The inequalities rank$(A) \leq n$ and rank$(A) \leq m$ hold.

(2) If the system is inconsistent, then rank$(A) < n$.

(3) If the system has exactly one solution, then rank$(A) = m$.

(4) If the system has infinitely many solutions, then rank$(A) < m$.

It is useful to consider the *contrapositives* of the statements in parts (b) through (d). What is the contrapositive of a statement? The contrapositive of the statement "if p then q" is "if not-q then not-p". For example, the contrapositive of the statement "If you live in New York City, then you live in the United States" is "If you do not live in the United States, then you do not live in New York City". A statement and its contrapositive are logically equivalent.

Example 2 Can you write down the contrapositive of the statements (b) through (d)?

(a) Contrapositive of (2) -

(b) Contrapositive of (3) -

(c) Contrapositive of (4) -

Number of Equations vs. Number of Unknowns

(1) If a linear system has exactly one solution, then there must be at least as many equations as there are variables ($m \leq n$ with the notation from Example 1).

Why?

Equivalently

(2) A linear system with fewer equations than unknowns ($n < m$) has either no solutions or infinitely many solutions.

Why? Hint: Think of two planes in space. Can they intersect at a point?

Example 3 Consider a linear system of n equations with n variables. When does this system have exactly one solution? Give your answer in terms of the rank of the coefficient matrix A.